Bubble-Flip – A New Generation Algorithm for Prefix Normal Words

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Prefix Normal Words

Prefix Normal Words

Definition

Prefix Normal Words

binary words

no factor has more 1s than the prefix of the same length

Fici, Lipták, "On prefix normal words". DLT ***[2011]***

w= 11010000110110 **not** prefix normal

There is a factor of **length 5** with >3 **1s.**

Prefix Normal Words

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#1s: 3 | 4

w= 110100011011 0 — **not** prefix normal

length: 5 | 5

u= 1101000100110 — **prefix normal**

Is there a factor of **length** 1 with >1 **1s?** no
Is there a factor of **length** 2 with >2 **1s?** no
Is there a factor of **length** 3 with >2 **1s?** no
⋮
Is there a factor of **length** 7 with >3 **1s?** no
⋮

Set of prefix normal words

$\mathcal{L}$: set of all finite prefix normal words

$\mathcal{L}_n$: set of all prefix normal words of length n

$\mathcal{L}_3$

000	*110*
100	*111*
101	

$\mathcal{L}_4$

0000	*1100*
1000	*1101*
1001	*1110*
1010	*1111*

$\mathcal{L}_5$

00000	*11001*
10000	*11010*
10001	*11011*
10010	*11100*
10100	*11101*
10101	*11110*
11000	*11111*

Basic Facts

Let $w \in \mathcal{L}$. Then

All prefixes of w are in $\mathcal{L}$

$w0 \in \mathcal{L}$

$\mathcal{L}$: set of all finite prefix normal words
$\mathcal{L}_n$: set of all finite prefix normal words of length n

#1s

length

$$w' = \overbrace{w_1 w_2 \cdots w_j w_{j+1}}^{k}_{j+1} \cdots \overbrace{w_{n-j+1} \cdots w_n 0}^{m \le k}_{j+1}$$

$w1 \in \mathcal{L}$ if and only if for all $1 \le j < n$, we have

$$|w_1 \cdots w_{j+1}|_1 > |w_{n-j+1} \cdots w_n|_1$$

$$w'' = \overbrace{w_1 w_2 \cdots w_j w_{j+1}}^{k}_{j+1} \cdots \overbrace{w_{n-j+1} \cdots w_n 1}^{> k}_{j+1}$$

A new generation algorithm

Generation algorithm

i.e., an algorithm for visiting each object in $\mathcal{L}_n$

Previous generation algorithm

- Runs in amortized $O(n)$ time per word
- Conjectured run in amortized $O(\log n)$ time per word, but still open

Burcsi, Fici, Lipták, Ruskey, Sawada, CPM ***[2014]***

New generation algorithm Bubble-Flip

- Runs in $O(n)$ time per word
- Gives **new insights into properties** of prefix normal words
- Makes steps towards answering conjecture from ***CPM [2014]***

Cicalese, Lipták, Rossi, LATA ***[2018]***

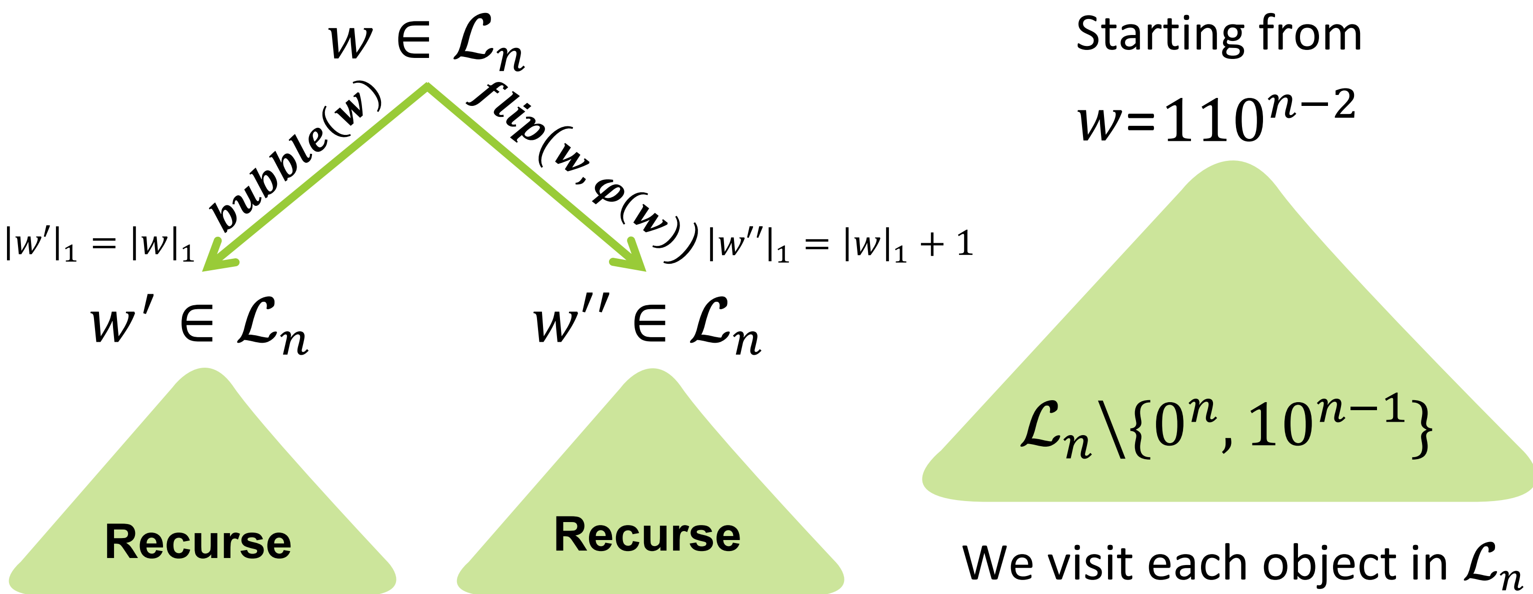
The Bubble-Flip Algorithm

Basic Idea

Generation algorithm visits each object in $\mathcal{L}_n$ exactly once

Based on two operations $\boldsymbol{bubble}(\boldsymbol{w})$ and $\boldsymbol{flip}(\boldsymbol{w}, \boldsymbol{\varphi}(\boldsymbol{w}))$

Recursive generation algorithm



Operation $bubble(w)$

$bubble(w)$: move rightmost 1 by one to the right

w= 11010001001**10**000 $\xrightarrow{bubble(w)}$ 11010001001**01**000

Lemma

Let $w \in \mathcal{L}_n$. Then $bubble(w) \in \mathcal{L}_n$

r Rightmost one

$$w = \overbrace{\underbrace{w_1 w_2 \cdots w_j}_{j}}^{k} \cdots \overbrace{\underbrace{w_{r-j-1} w_{r-j} \cdots w_{r-1}}_{j}}^{m \leq k} 1000000$$

$bubble(w)$

$$w' = \overbrace{\underbrace{w_1 w_2 \cdots w_j}_{j}}^{k} \cdots w_{r-j-1} \overbrace{\underbrace{w_{r-j} \cdots w_{r-1} 0}_{j}}^{\leq m \leq k} 100000$$

Operation $flip(w, \varphi(w))$

Phi $\boldsymbol{\varphi}(w)$: Minimum position j after the rightmost 1 such that $flip(w, j)$ is prefix normal (if such a j exists)

Rightmost one r

w = 1101000100110000 $\xrightarrow{flip(w,r+1)}$ 1101000100111000

w = 1101000100110000 $\xrightarrow{flip(w,r+2)}$ 1101000100110100

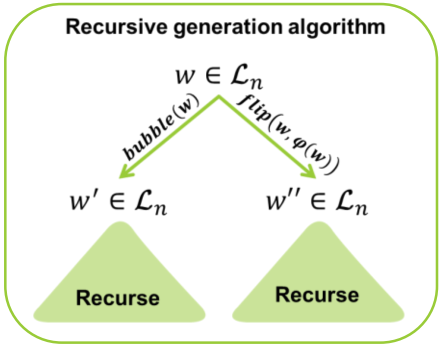
w = 1101000100110000 $\xrightarrow{flip(w,r+3)}$ 1101000100110010

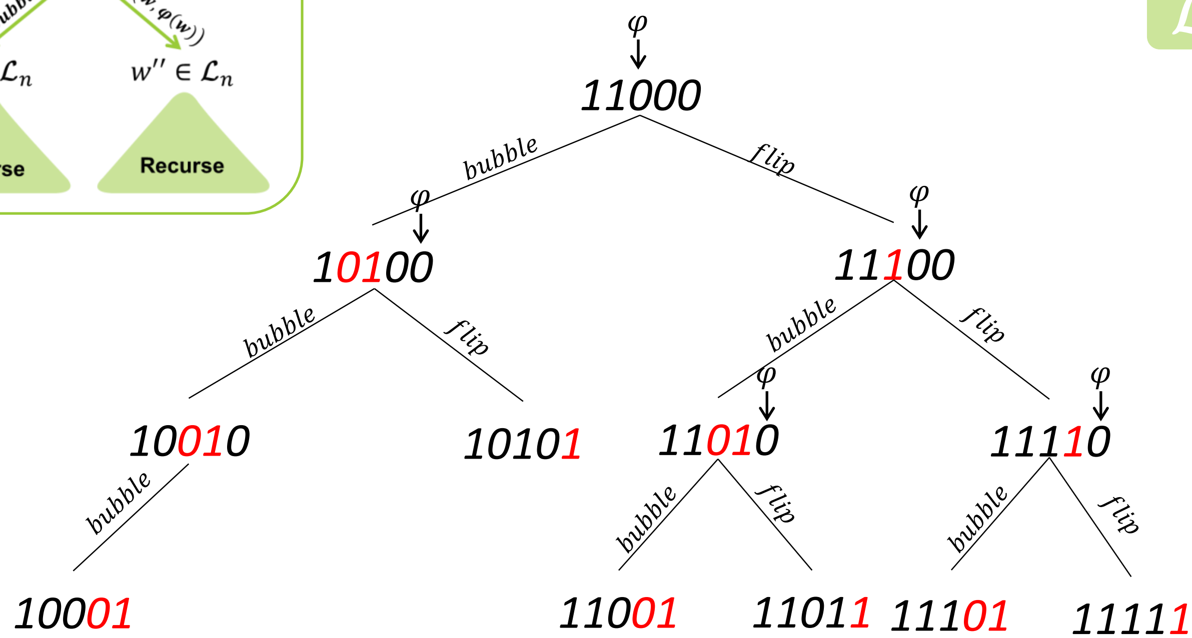
$$\varphi(w) = r + 3$$

Fact

Let $w \in \mathcal{L}_n$. Then $flip(w, \varphi(w)) \in \mathcal{L}_n$ $(by\ definition\ of\ \varphi(w))$

Bubble-Flip algorithm





In-order traversal

$\mathcal{L}_5$

00000
10000
10001
10010
10100
10101
11000
11001
11010
11011
11100
11101
11110
11111

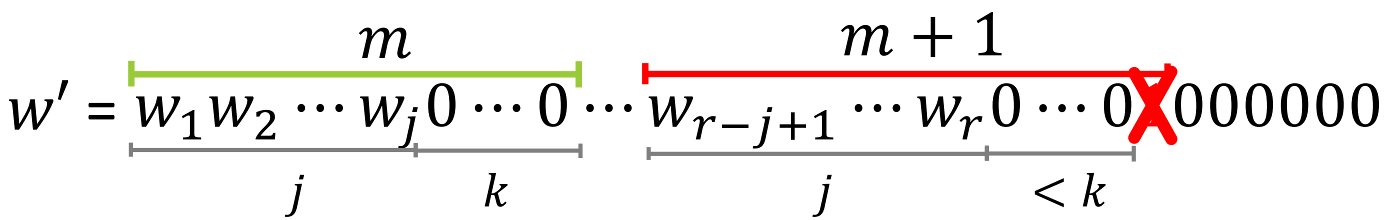
The in-order traversal visits the objects in lexicographic order

The post-order traversal generates a **Gray code**

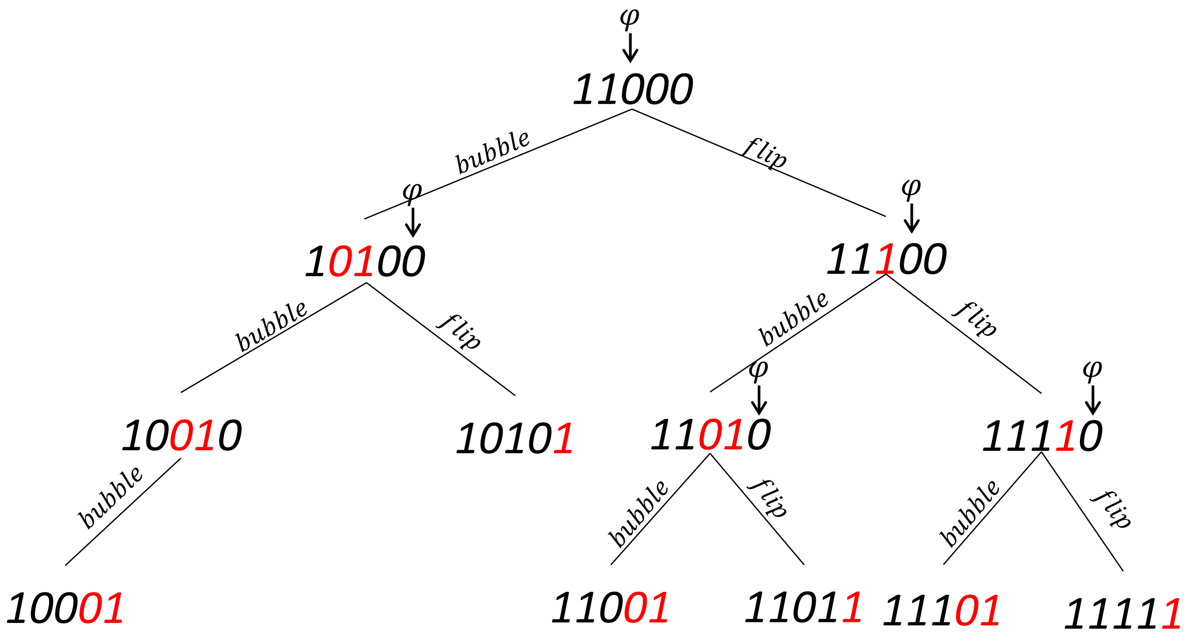
Computation of φ

Phi $\boldsymbol{\varphi}(w)$: Minimum position j after the rightmost 1 such that $flip(w, j)$ is prefix normal (if such a j exists)

- **Easy** in quadratic time
- Can be computed in **linear** time:



Analysis of Bubble-Flip algorithm



Computation cost (without outputting) **per word:**

- $bubble(w)$: $O(1)$
- $\varphi(w)$: $\boldsymbol{O(n)}$
- $flip\big(w, \varphi(w)\big)$: $O(1)$

Infinite prefix normal words

Infinite prefix normal words and operation flipext

Definition

Infinite Prefix Normal Words

infinite binary words

no factor has more 1s than the prefix of the same length

Let $w \in \mathcal{L}$. Then $w0^{\omega}$ is an **infinite prefix normal word**

Operation $flipext(w)$ — *extension version of* $flip(w, \varphi(w))$

Let $w \in \mathcal{L}$. $flipext(w)$ is the finite word $w0^k1$, where $k = \min\{j \mid w0^j1 \in \mathcal{L}\}$.

The infinite word $v = flipext^{\omega}(w) = \lim_{i \to \infty} flipext^{(i)}(w)$

w= 110100010011 $\xrightarrow{flipext^{(1)}(w)}$ 110100010011001

1101000100110010010011001001001100100100110010 01

Minimum density

Let $w \in \{0,1\}^* \cup \{0,1\}^\omega$. Then

Density

$$D(i) = \frac{|w_1 \cdots w_i|_1}{i}$$

Minimum density

$$\rho(w) = inf\{D(i) | 1 \leq i\}$$

Iota *(minimum density prefix)*

$$\iota(w) = \min\{j \mid \forall i : D(j) \leq D(i)\}$$ if it exists

$\kappa(w) = 4$

$w =$ 1101000100 11

$\iota(w) = 10$

Minimum density

$\rho(w) = \frac{4}{10}$

v — 10101010⋯ $\rho(v) = \frac{1}{2}$, $\iota(v) = 2$

u — 110101010⋯ $\rho(u) = \frac{1}{2}$, $\iota(u)$ is undefined

Results on infinite words generated by $flipext^{\omega}(w)$

Let $w \in \mathcal{L}$ and $v = flipext^{\omega}(w)$. Then

$\boldsymbol{\rho(v) = \rho(w)}$, and as consequence $\boldsymbol{\iota(v) = \iota(w)}$ and $\boldsymbol{\kappa(v) = \kappa(w)}$

v is the **densest** prefix normal word with $\boldsymbol{w}$ as prefix

v is **ultimately periodic**. In particular,

- v can be written as $\boldsymbol{v = ux^{\omega}}$

where $|x| = \iota(w)$, $|x|_1 = \kappa(w)$ and x is prefix normal.

Moreover if x not a suffix of u

- $|u| \leq \left(\binom{\iota}{\kappa} - 1\right) m\, \iota$

where $\iota = \iota(w)$, $\kappa = \kappa(w)$ and $m = \left\lfloor \frac{|w|}{\iota} \right\rfloor$

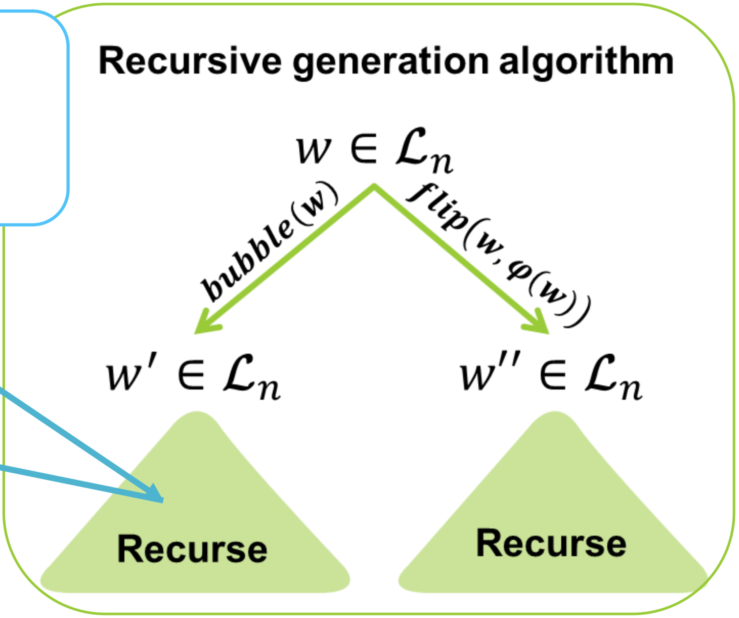
Open problems - **Generation algorithm**

- **sublinear** time per word?
- Can we find $\varphi\left(flip(w,\varphi(w))\right)$ in sublinear time?
  - We can find $\varphi(bubble(w))$ in $O(1)$

How many nodes are there**?**

How many prefix normal words of length n with a given prefix are there**?**

TCS ***[2017]***



Open problems – Infinite Words

- Given $\boldsymbol{w}$ we can generate $\boldsymbol{flipext^{\omega}(w) = ux^{\omega}}$:
  - We can find $\boldsymbol{x}$ in $\boldsymbol{O}\left(|\boldsymbol{w}| \cdot \left|\boldsymbol{ux^{m+1}}\right|_{\mathbf{1}}\right)$ where $m = \left\lfloor \frac{|w|}{\iota(w)} \right\rfloor$
  - Can we find $\boldsymbol{x}$ faster? i.e.: $\boldsymbol{O}(|\boldsymbol{w}|)$

We have reasons to belive that this is possible

- There exist infinite prefix normal words for all $\boldsymbol{\rho(v)} \in \mathbb{R}$:
  - $\boldsymbol{v}$ is **ultimately periodic** then $\boldsymbol{\rho(v)}$ is **rational**
  - $\boldsymbol{\rho(v)}$ is **irrational** then $\boldsymbol{v}$ is **aperiodic**
- Are there any other periodicity properties?
- Are there any other properties about infinite words?

THANK YOU!

Iota-factorization

Iota-factorization of w

Is the factorization of w into ι-length factors.

$w = u_1 u_2 \cdots u_r v$ and where $r = \lfloor |w|/\iota \rfloor$, $|u_i| = \iota$ for $i = 1, \ldots, r$, and $|v| < \iota$, for w finite,

$w = u_1 u_2 \cdots$ and where $|u_i| = \iota$ for all i, for w infinite.

$\iota(w) = 10$	$\iota(w)$	$\iota(w)$	$\iota(w)$	$\iota(w)$
1101000100	1100100100	1100100100	1100100100	1100100100
$\kappa(w) = 4$	$\kappa(w)$	$\kappa(w)$	$\kappa(w)$	$\kappa(w)$

Lemma

Let w be a finite or infinite prefix normal word, such that $\iota = \iota(w)$ exists.
Let $w = u_1 u_2 \cdots$ be the Iota-factorization of w. **Then for all $\boldsymbol{i}$, $\boldsymbol{u_i \geq_{lex} u_{i+1}}$.**

Iota-factorization properties

Lemma

Let w be a finite or infinite prefix normal word, such that $\iota = \iota(w)$ exists. Let $w = u_1 u_2 \cdots$ be the Iota-factorization of w. **Then for all $\boldsymbol{i}$, $\boldsymbol{u_i \geq_{lex} u_{i+1}}$.**

$$w = \quad \cdots \; \underbrace{a_1 a_2 \cdots a_j \overbrace{a_{j+1} \cdots a_\iota}}_{u_i} \underbrace{\overbrace{b_1 b_2 \cdots b_j}^{\leq \kappa} b_{j+1} \cdots b_\iota}_{u_{i+1}} \cdots$$

Let $\mathrm{h} = \min\{\mathrm{j} \mid |a_{j+1} \cdots a_\iota b_1 b_2 \cdots b_j|_1 < \kappa\}$ then it holds that $a_h > b_h$

Since two consecutive factors have the same prefix, the first difference must be a 1 that turns into a 0. Then $\boldsymbol{u_i \geq_{lex} u_{i+1}}$.