UNIVERSITÀ di VERONA

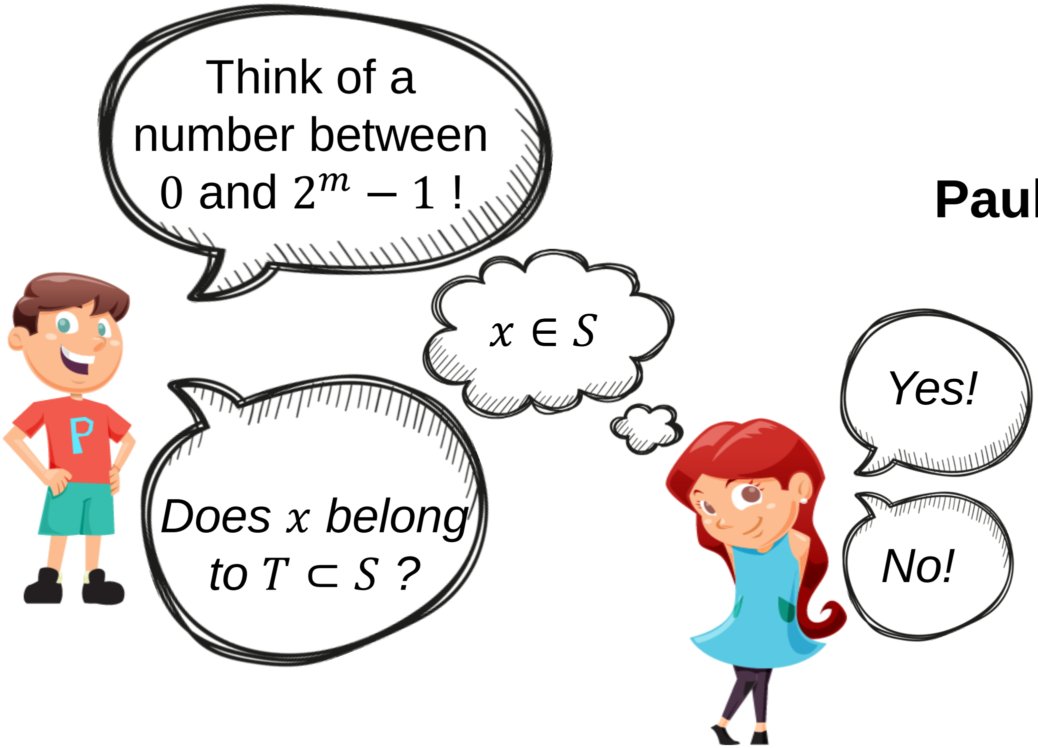
Dipartimento di INFORMATICA

On the multi-interval Ulam-Rényi game: For 3 lies 4 intervals suffice

*Ferdinando Cicalese and **Massimiliano Rossi***
Departement of Computer Science - University of Verona

Problem Statement

The rules of the game:



Paul and **Carole** fix a search space $S = \{0, \dots, 2^m - 1\}$ and an integer $e \geq 0$.

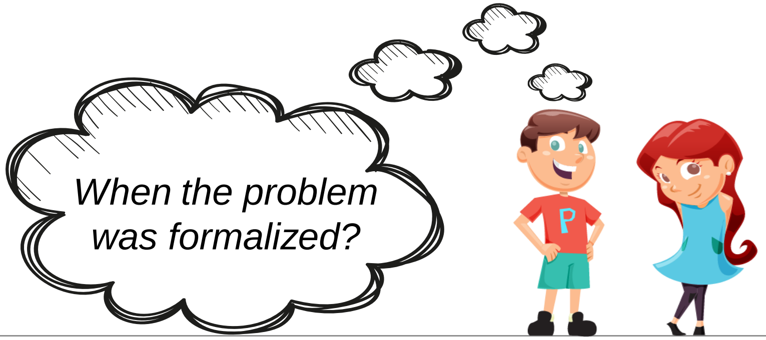
Carole chooses a number $x \in S$.

Paul must guess x by asking questions of the form *"does x belong to $T \subset S$?".*

Carole can answer only *"yes"* or *"no".*

Carole can lie at most e times.

History



[1961] *«… when only two things can be thought of and only one lie is allowed, then 3 questions are needed…» Rényi, A.*

"On a problem of information theory"

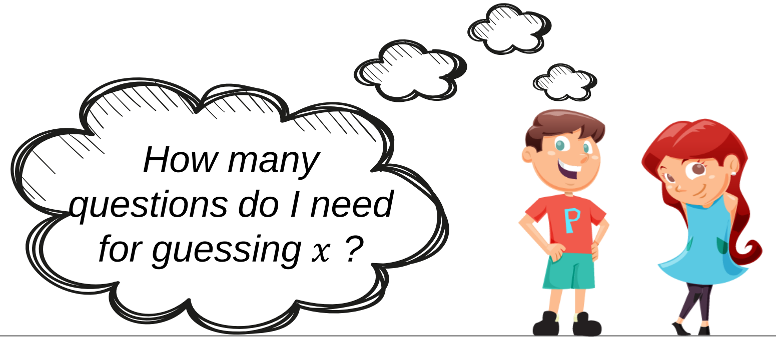
[1968] *«… at each stage the questioned set may depend on the entire past history of the game …» Berlekamp, E. R.*

"Block coding with noiseless feedback"

[1976] *«… One clearly needs more than n questions for guessing one of the 2^n objects because one does not know when the lie was told…» Ulam, S. M.*

"Adventures of a Mathematician"

State of the art



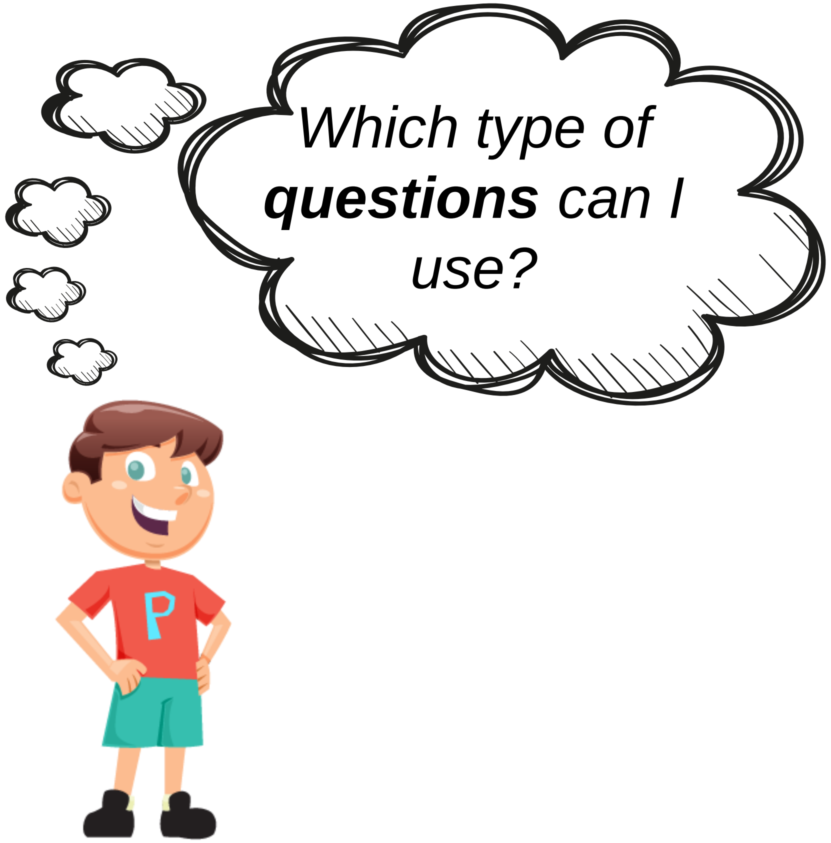
[1968] *Berlekamp, E.R.* Lower Bound: $min\left\{q \mid \sum_{j=0}^{e}\binom{q}{j} \leq 2^{q-m}\right\}$

Number of lies (e)	**Solution for $S = \{0, \dots, 2^m - 1\}$**
0	$Binary\ Search$
1	***[1987]*** *Pelc, A.*
2	***[1989]*** *Czyzowicz, J., Mundici, D. Pelc, A.*
3	***[1992]*** *Negro, A. , Sereno, M.*
e	***[1992]*** *Spencer, J.*

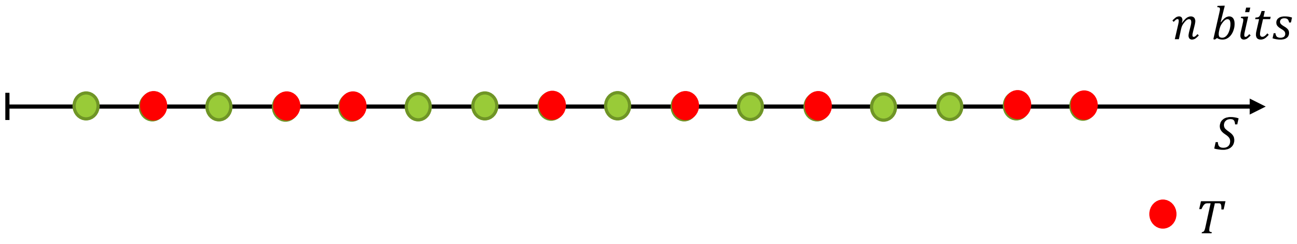
Definition

Paul's **strategy** is **perfect** if it guarantees him to identify Carole's number x using at most the number of questions given by Berlekamp's **Lower Bound**.

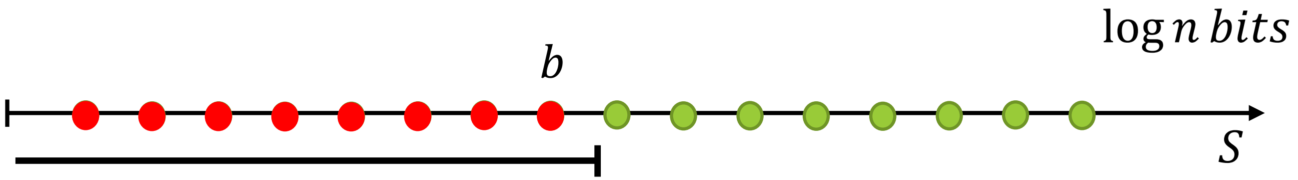
Not all subsets available



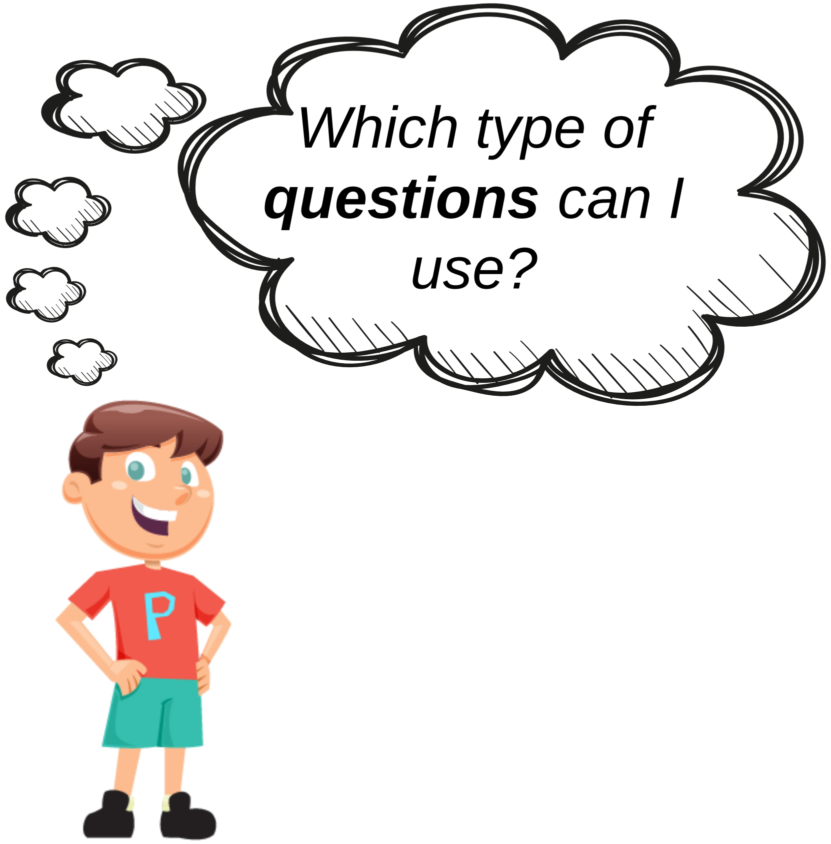
Arbitrary *yes-no* questions “does x belong to $T \subset S$?”



Comparison questions “Is $x \leq b$?”

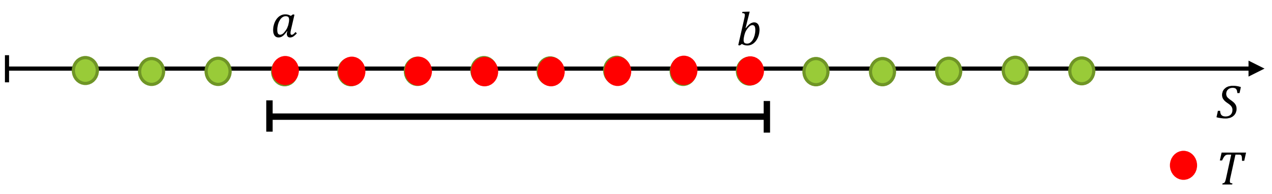


Not all subsets available



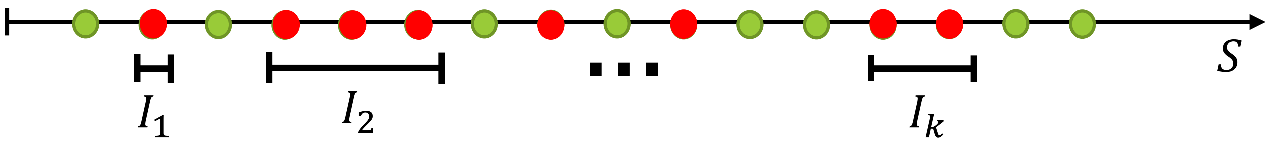
Interval questions "Is $a \leq x \leq b$?"

$2\log n \; bits$

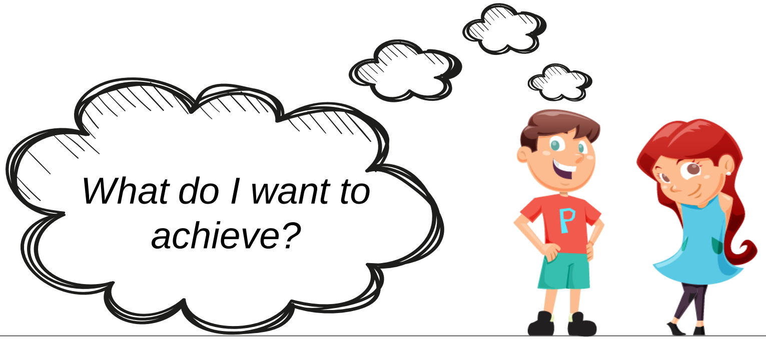


Multi-Interval questions "does x belong to $I_1 \cup I_2 \cup \cdots \cup I_k$?"

$2k \log n \; bits$



Main objective



Definition

Paul's **strategy** is **perfect** if it guarantees him to identify Carole's number x using at most the number of questions given by Berlekamp's **Lower Bound**.

Objective

Fixed the number of lies $e \geq 0$.
We want to find the **minimum k** such that there exists a **perfect strategy** which uses only **k-intervals questions**.

State of the art

Definition

Paul's **strategy** is **perfect** if it guarantees him to identify Carole's number x using at most the number of questions given by Berlekamp's **Lower Bound**.

Objective

Fixed the number of lies $e \geq 0$.
We want to find the **minimum k** such that there exists a **perfect strategy** which uses only **k-intervals questions**.

Number of lies (e)	Number of intervals	Solution $S = \{0, \dots, 2^m - 1\}$
0	$1 \,/\, 2$	$Binary\ Search$
1	1	***[2013]*** *Cicalese, F.*
2	2	***[1997]*** *Mundici, D. Trombetta, A.*
$\vdots$	$\vdots$	
e	$\frac{e^2+3e+2}{2}$	***[2013]*** *Cicalese, F.*

Main Result

Definition

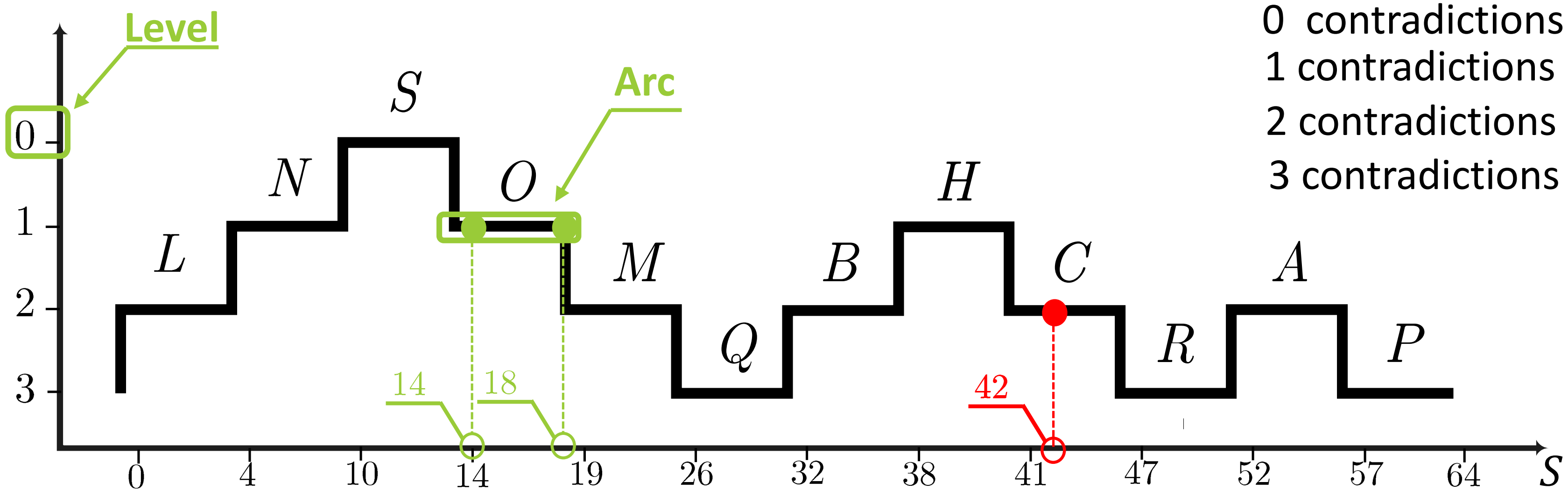
Paul’s **strategy** is **perfect** if it guarantees him to identify Carole’s number x using at most the number of questions given by Berlekamp’s **Lower Bound**.

Objective

Fixed the number of lies $e \geq 0$.
We want to find the **minimum k** such that there exists a **perfect strategy** which uses only **k-intervals questions**.

Number of lies (e)	Number of intervals	Solution $S = \{0, \dots, 2^m - 1\}$
0	1 / 2	*Binary Search*
1	1	***[2013]*** *Cicalese, F.*
2	2	***[1997]*** *Mundici, D. Trombetta, A.*
3	4	***[2017]*** *Cicalese, F. Rossi, M.*
e	$\frac{e^2+3e+2}{2}$	***[2013]*** *Cicalese, F.*

State Representation



0 contradictions: S

1 contradictions : $N \cup O \cup H$

2 contradictions : $L \cup M \cup B \cup C \cup A$

3 contradictions : $Q \cup R \cup P$

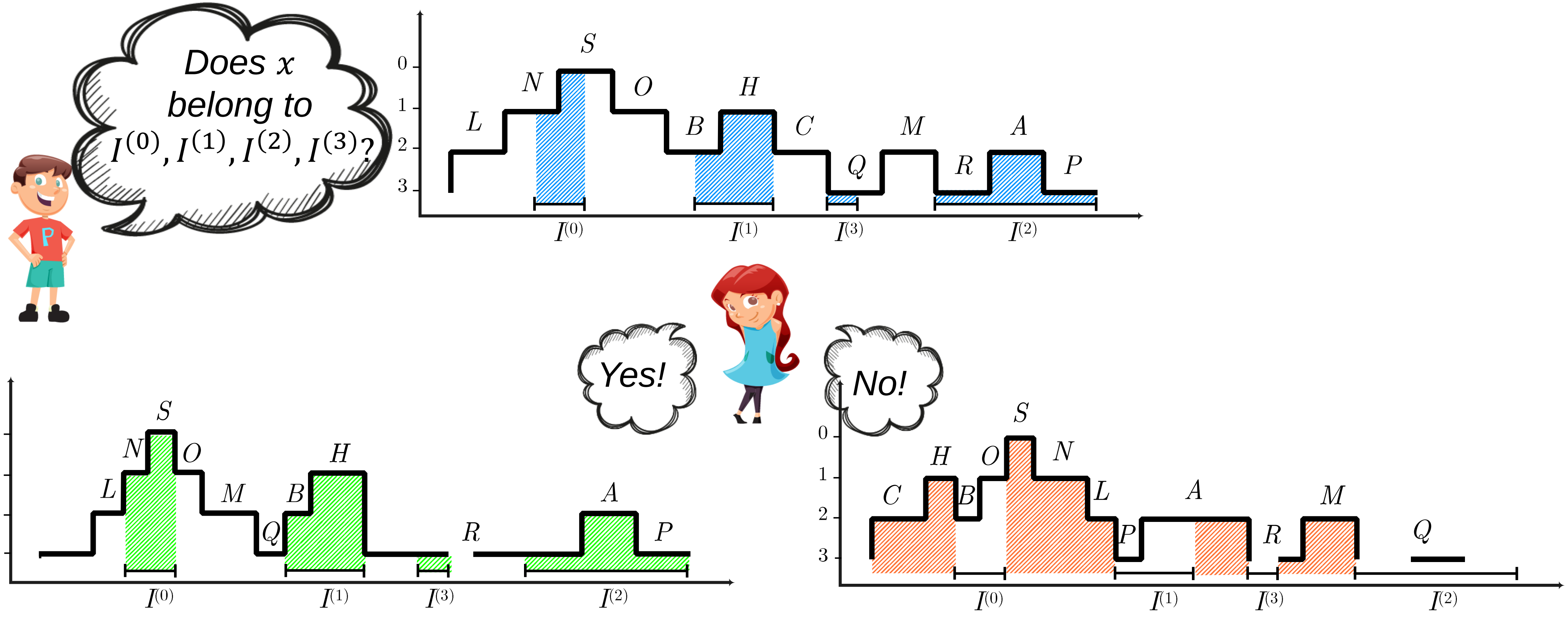
Structure Property:

Every arc has neighboring arcs at exactly **one** level of distance.

Well-Shaped State: a state σ is well shaped iff

- For $i = 0, \cdots, e-1$, σ has exacly $2i+1$ arcs lying on level i.
- σ has exacly e arcs lying on level e.

State Dynamics



Key Result

Definition

Given a state σ, let $ch(\sigma)$ be the minimum number of questions needed to find x according to the *Berlekamp's* **Lower Bound**.

Definition

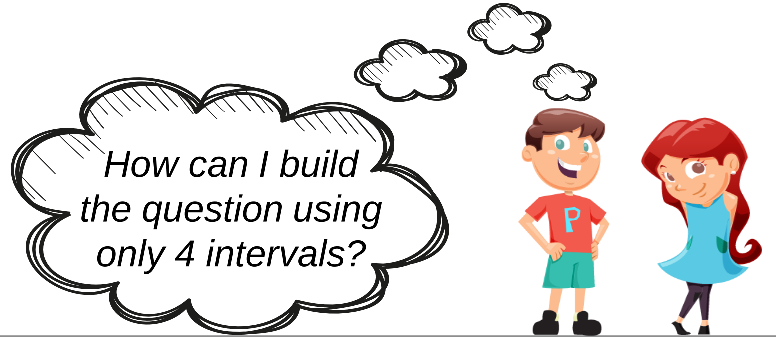
A state σ is **final** if it contains only one element, and $ch(\sigma) = 0$

Theorem

Given a well-shaped state σ there is a **4-interval question** such that the resulting states σ_{yes} and σ_{no} are well shaped and

$$\max\{ch(\sigma_{yes}), ch(\sigma_{no})\} < ch(\sigma)$$

Question Construction



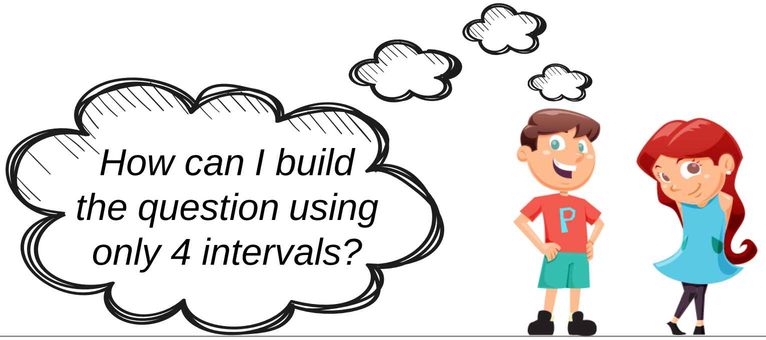
Intuition

To keep $\max\{ch(\sigma_{yes}), ch(\sigma_{no})\} < ch(\sigma)$ we **split evenly** each level between *yes* and *no*.

Intuition

To keep σ_{yes} and σ_{no} well-shaped we let the question **split** at most one arc per level.

Question Construction

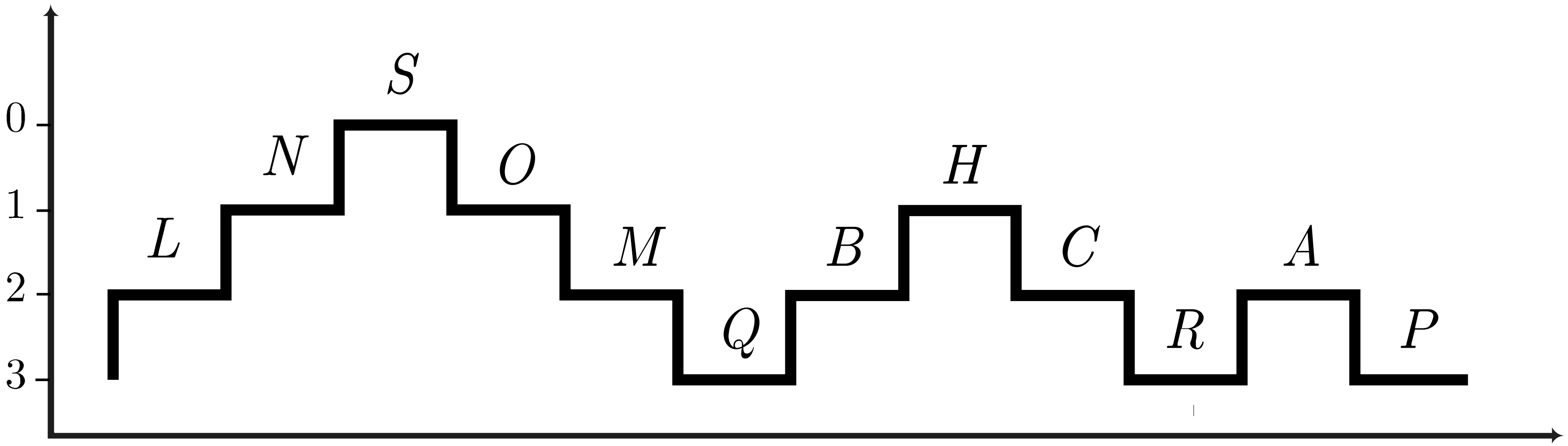


Level 0

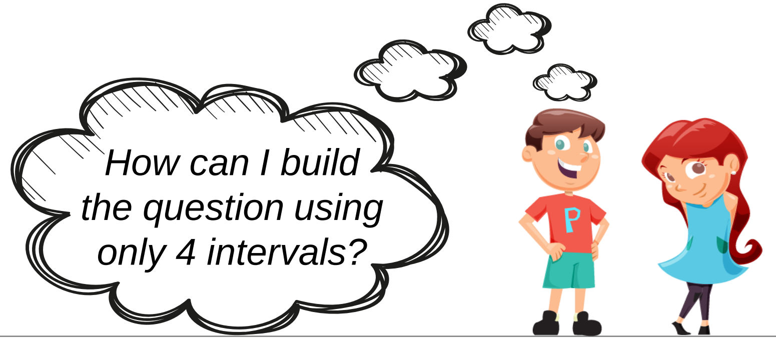
Level 1

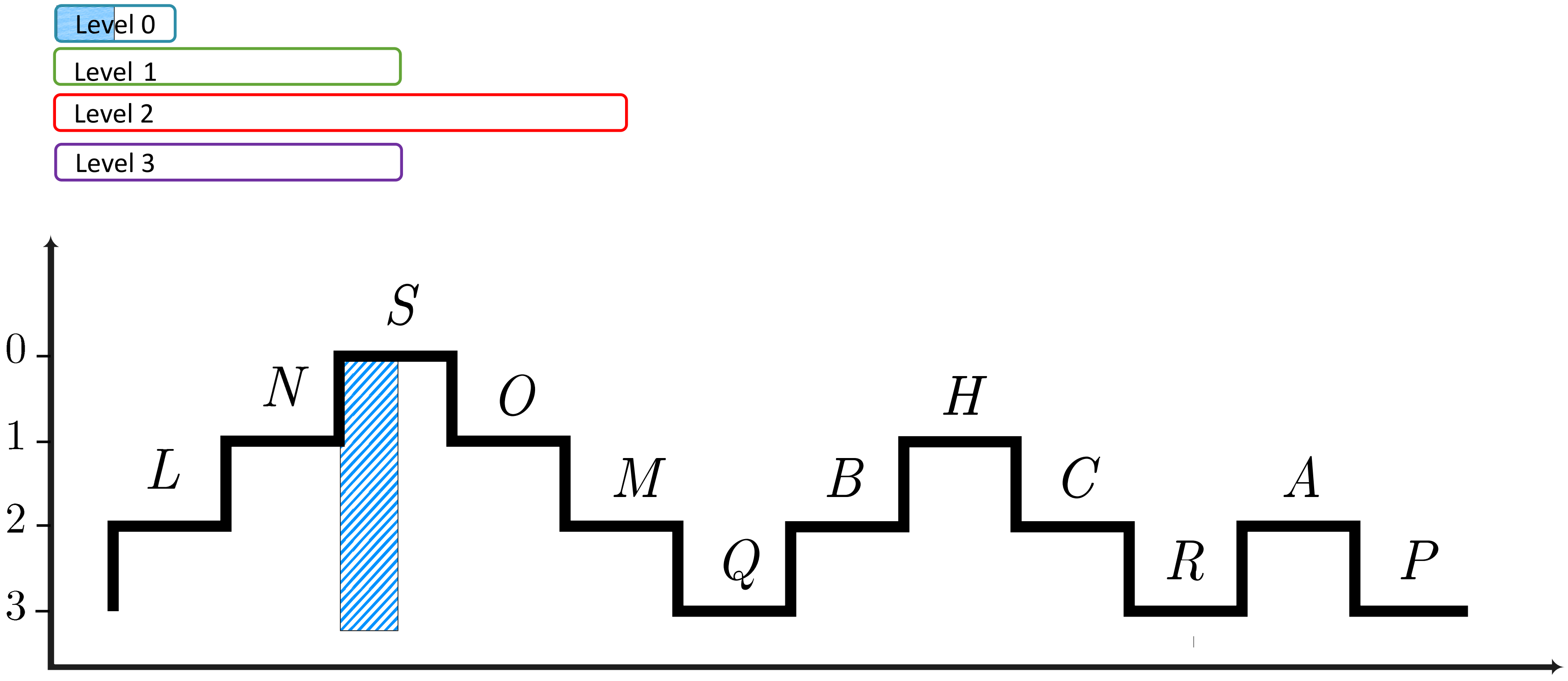
Level 2

Level 3

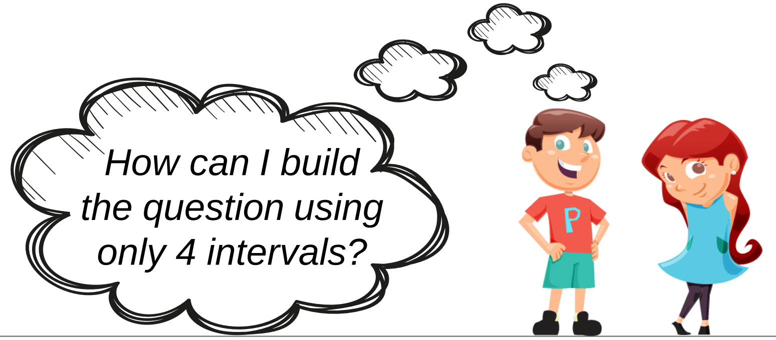


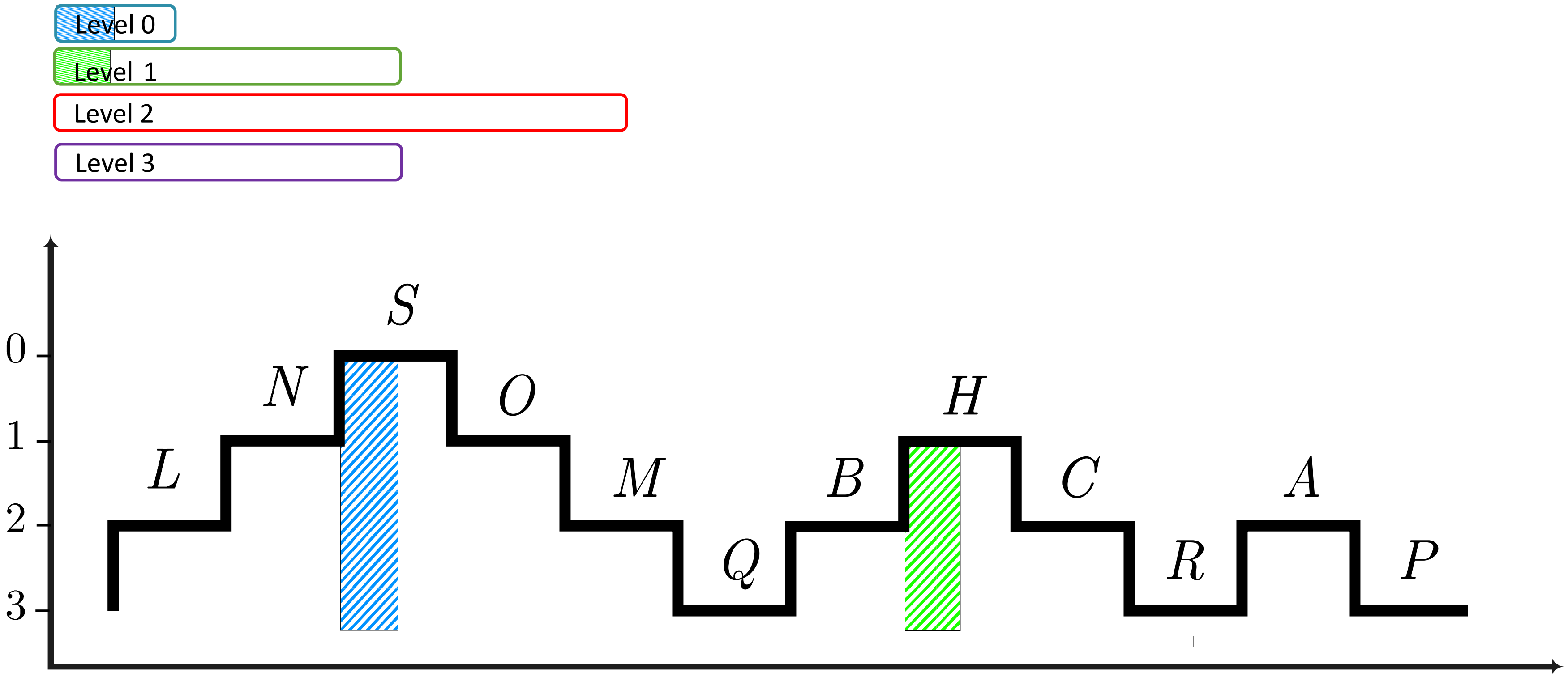
Question Construction



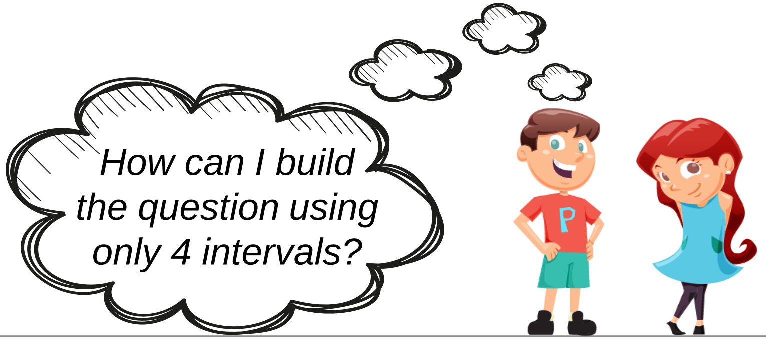


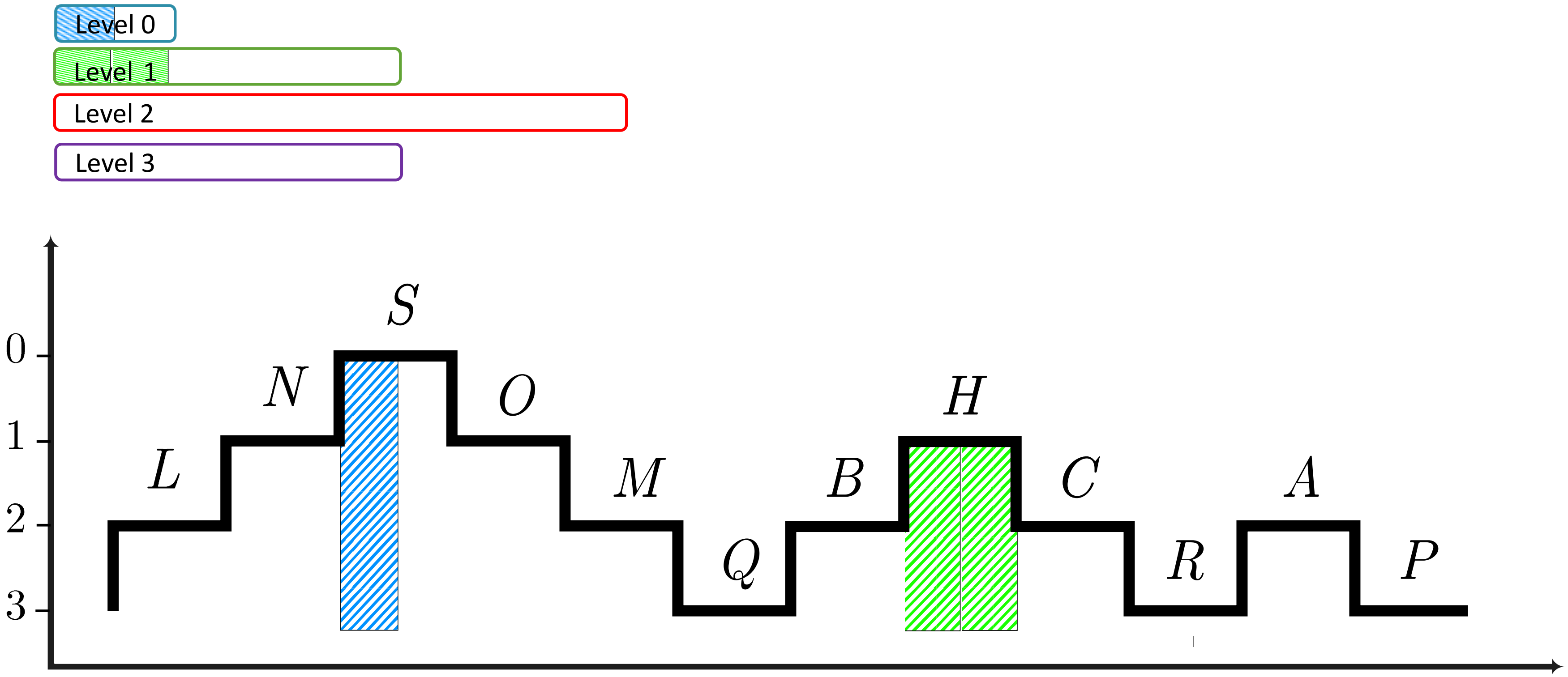
Question Construction



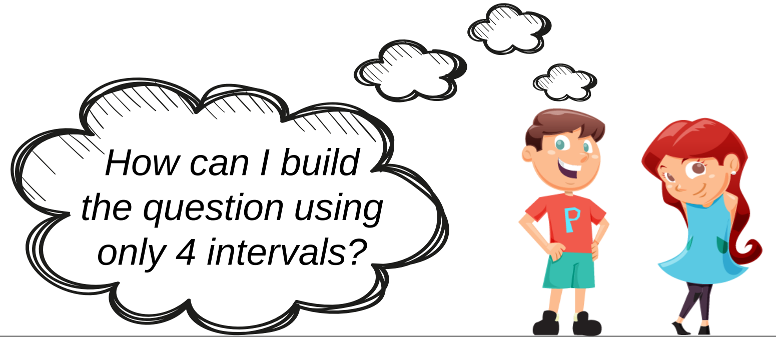


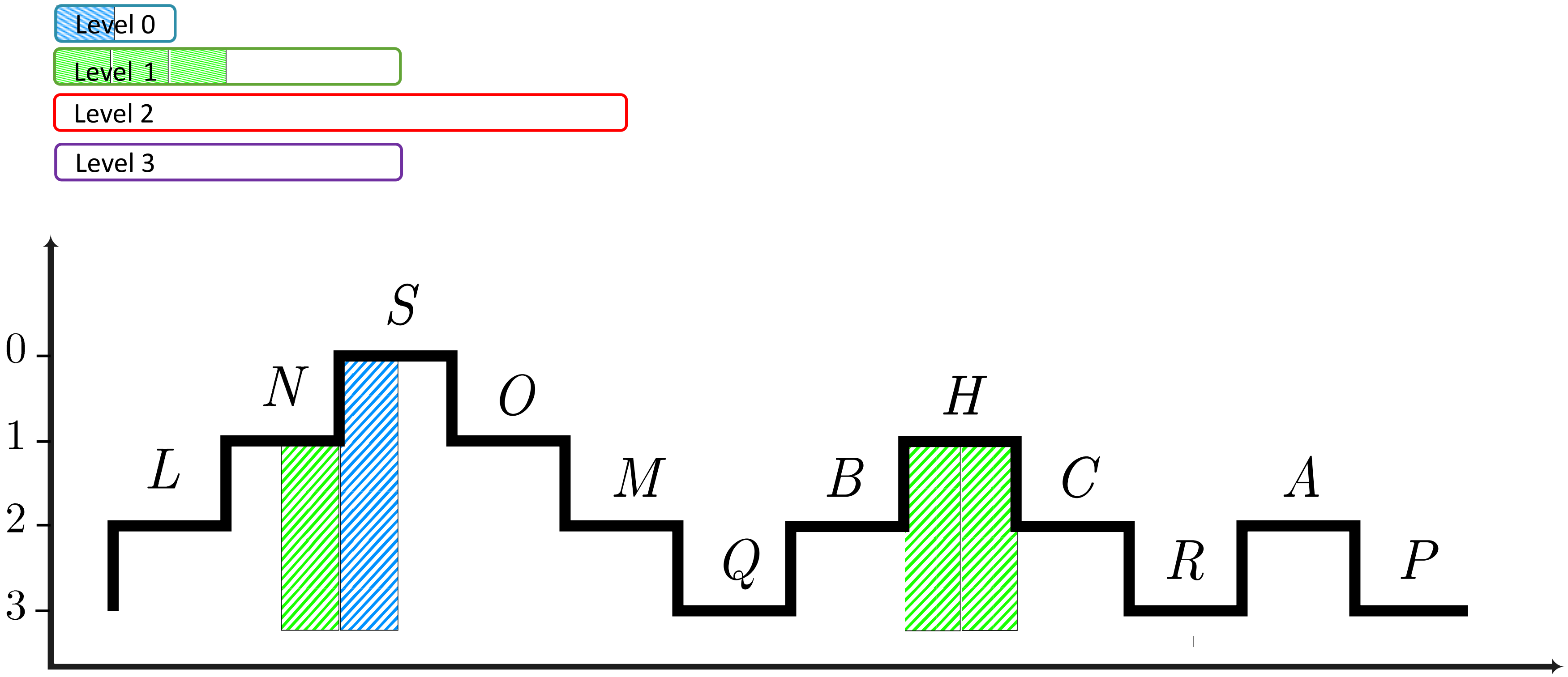
Question Construction



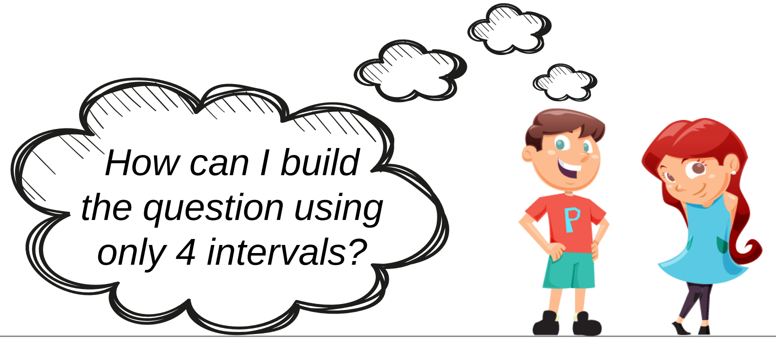


Question Construction





Question Construction

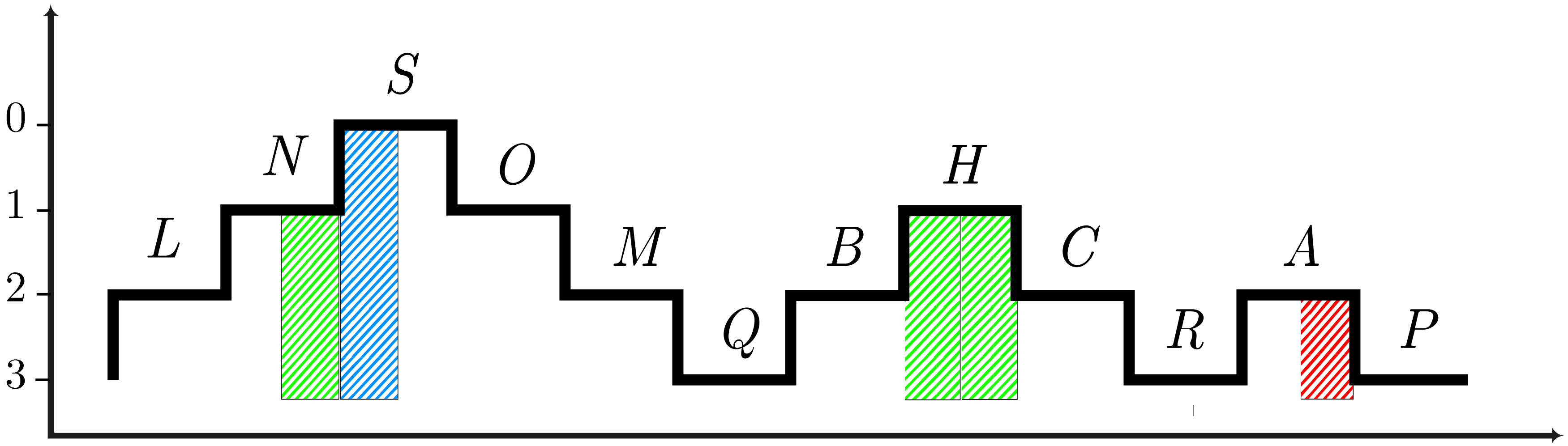


Level 0

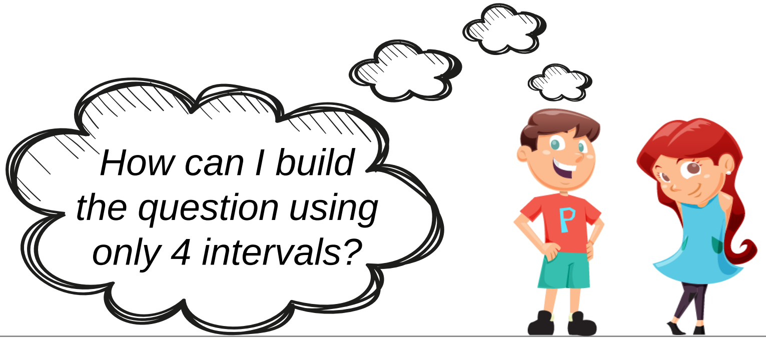
Level 1

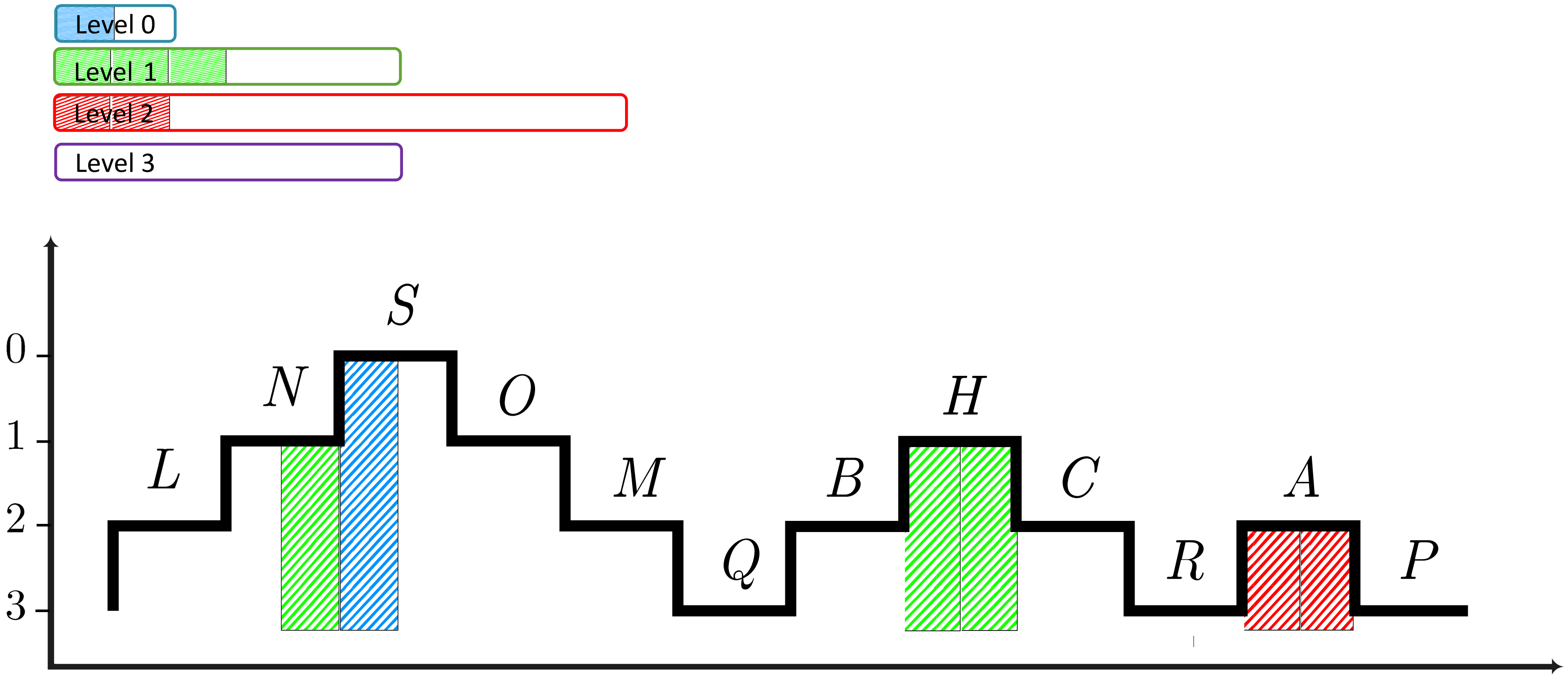
Level 2

Level 3

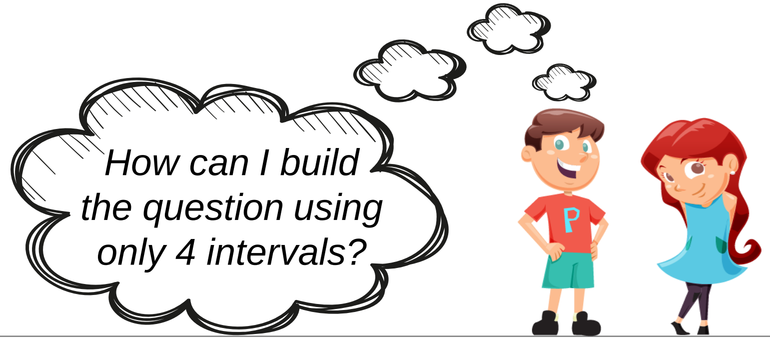


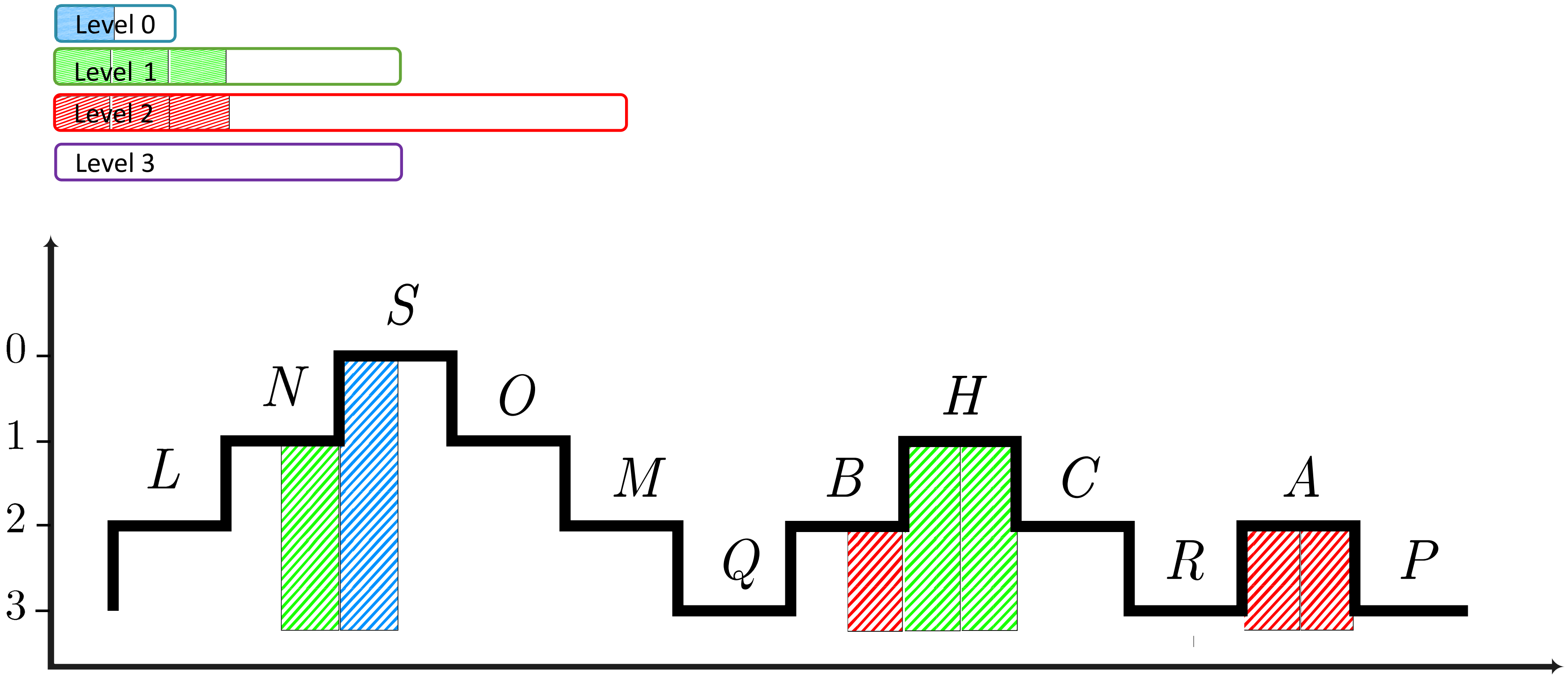
Question Construction



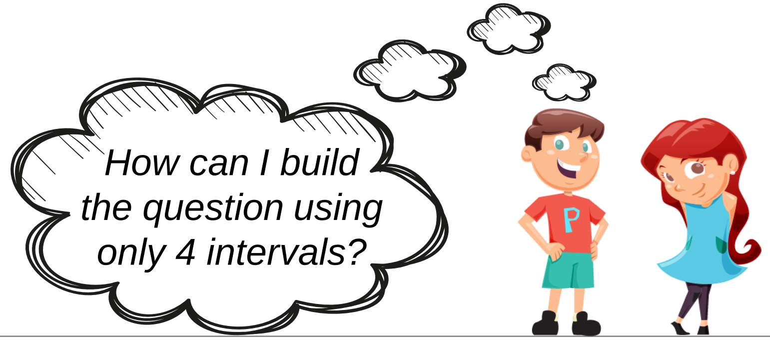


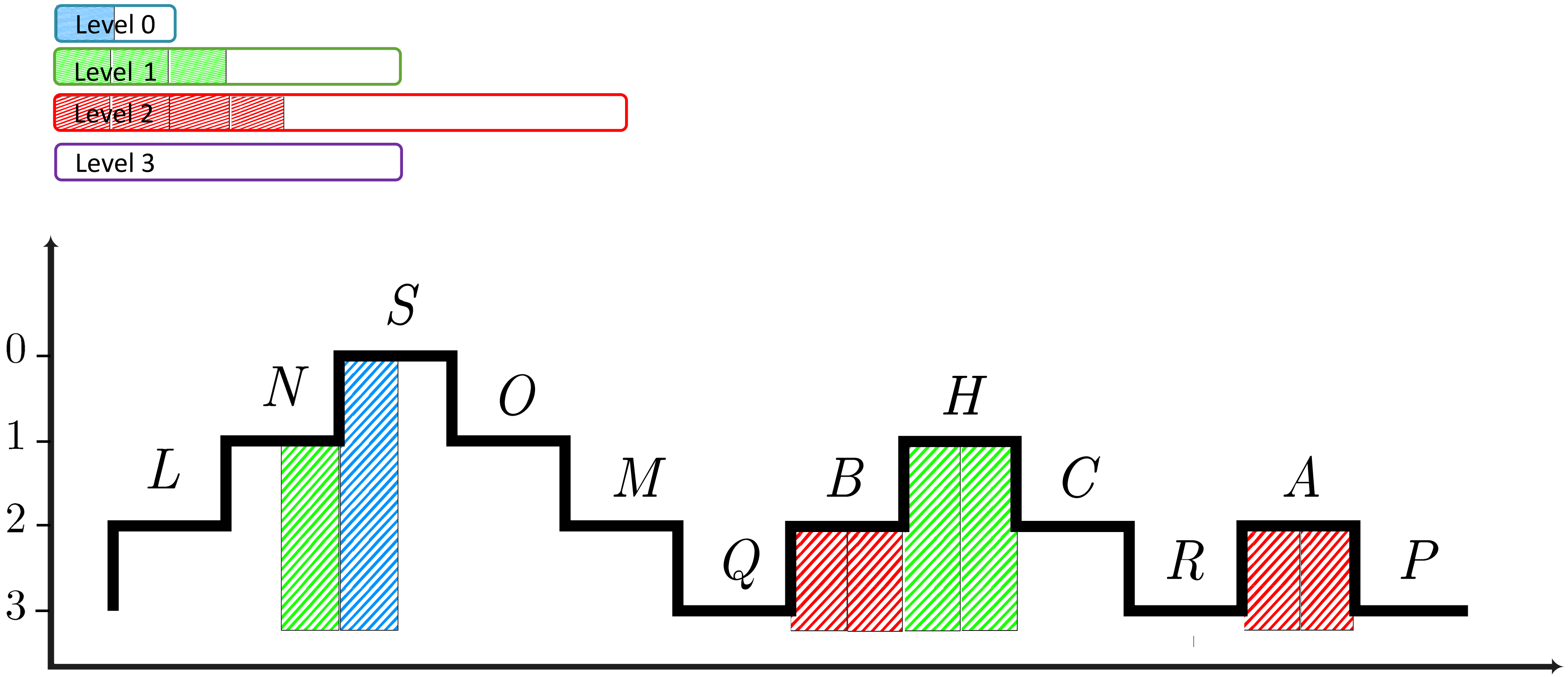
Question Construction



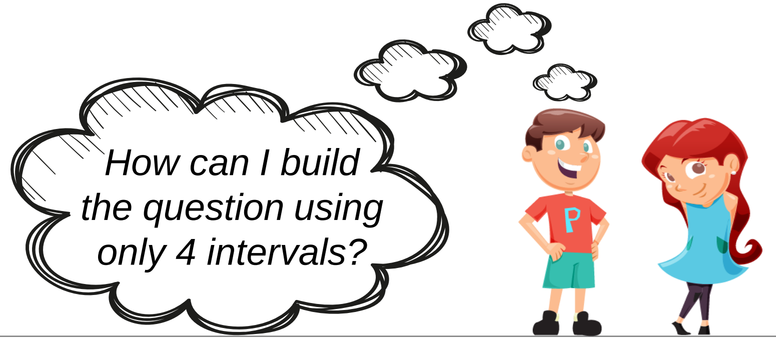


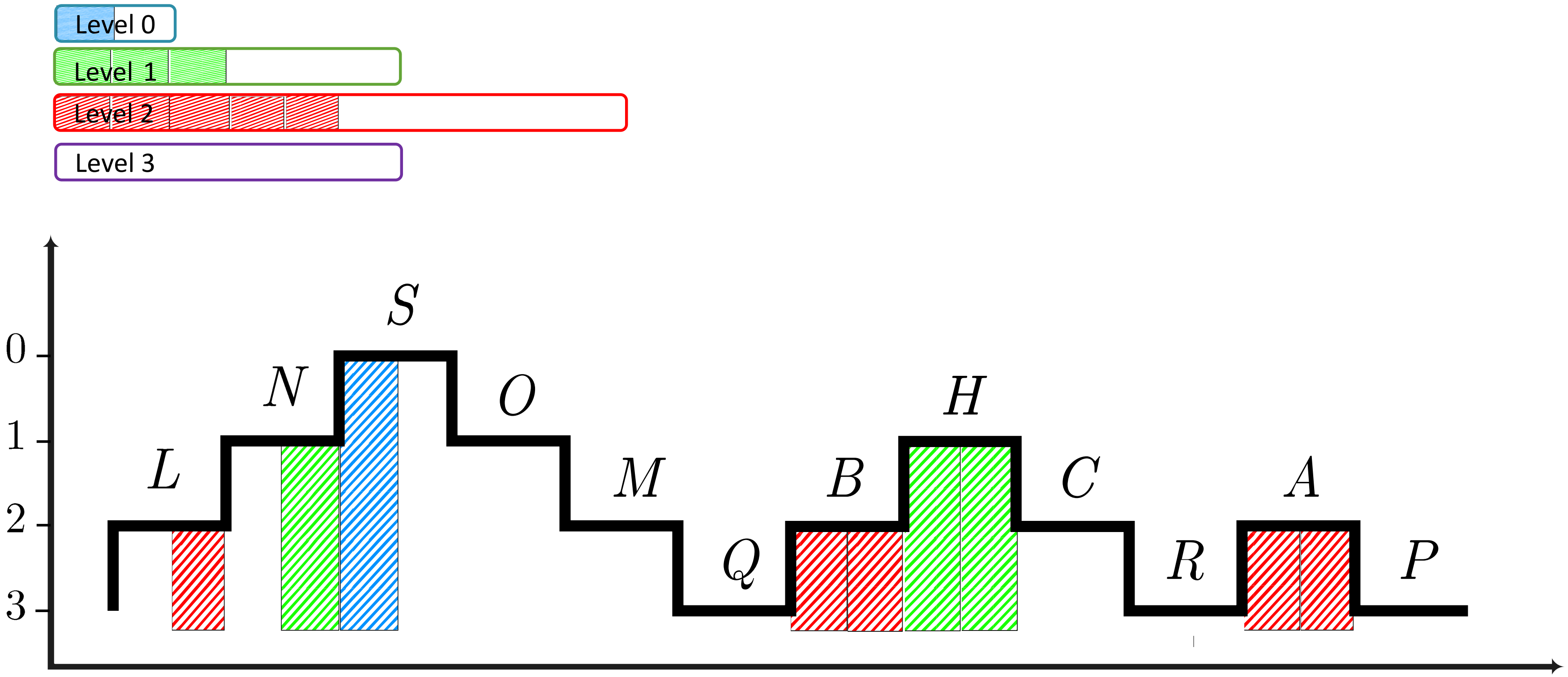
Question Construction



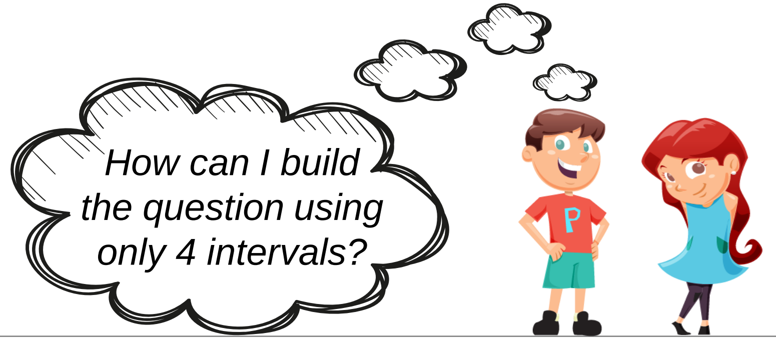


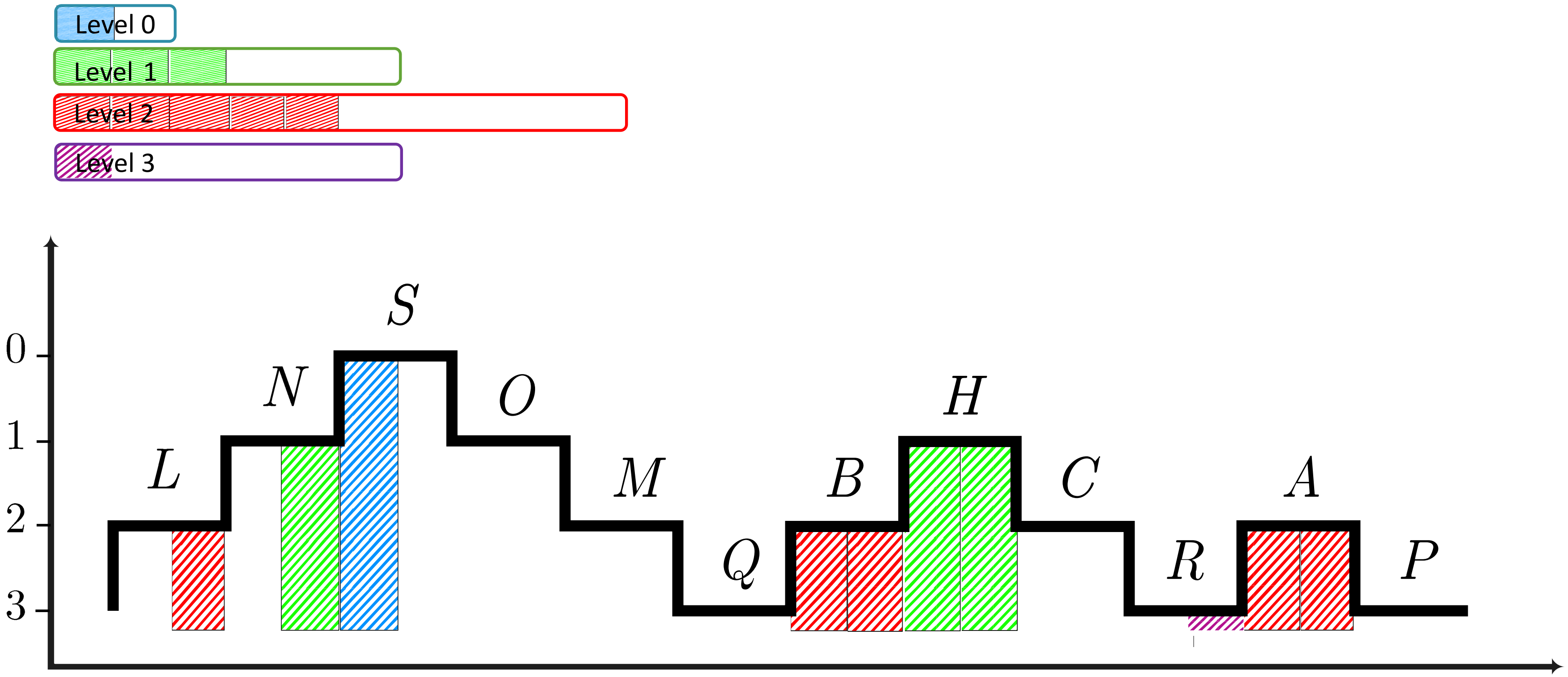
Question Construction



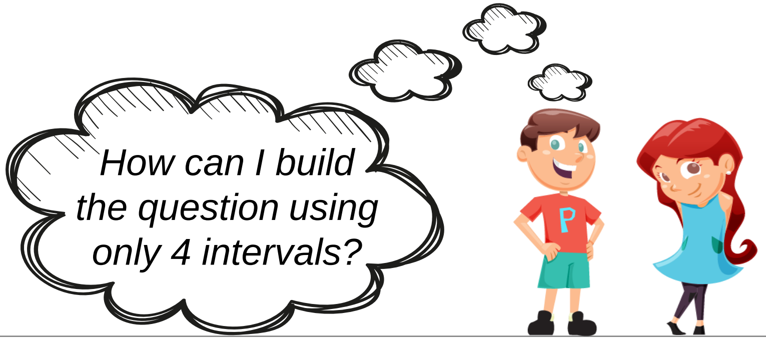


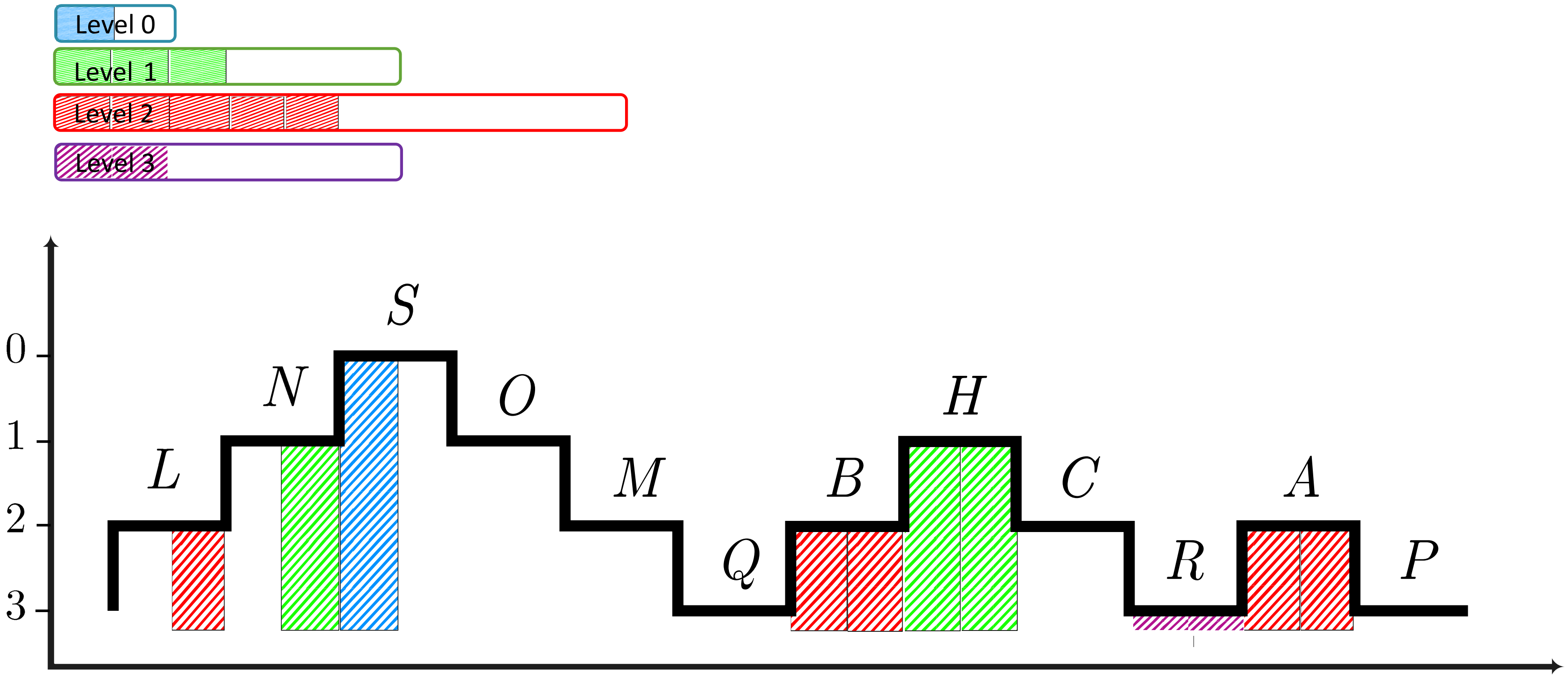
Question Construction



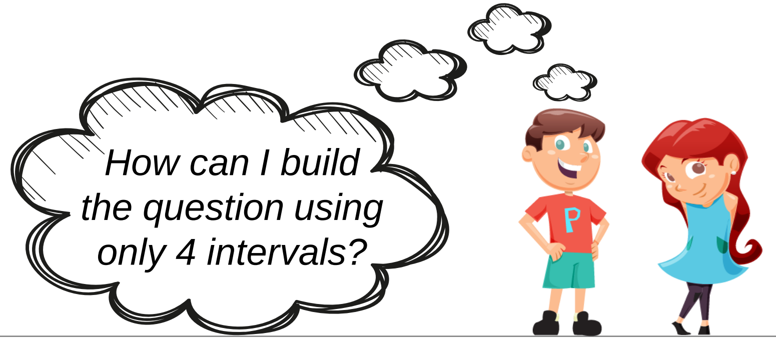


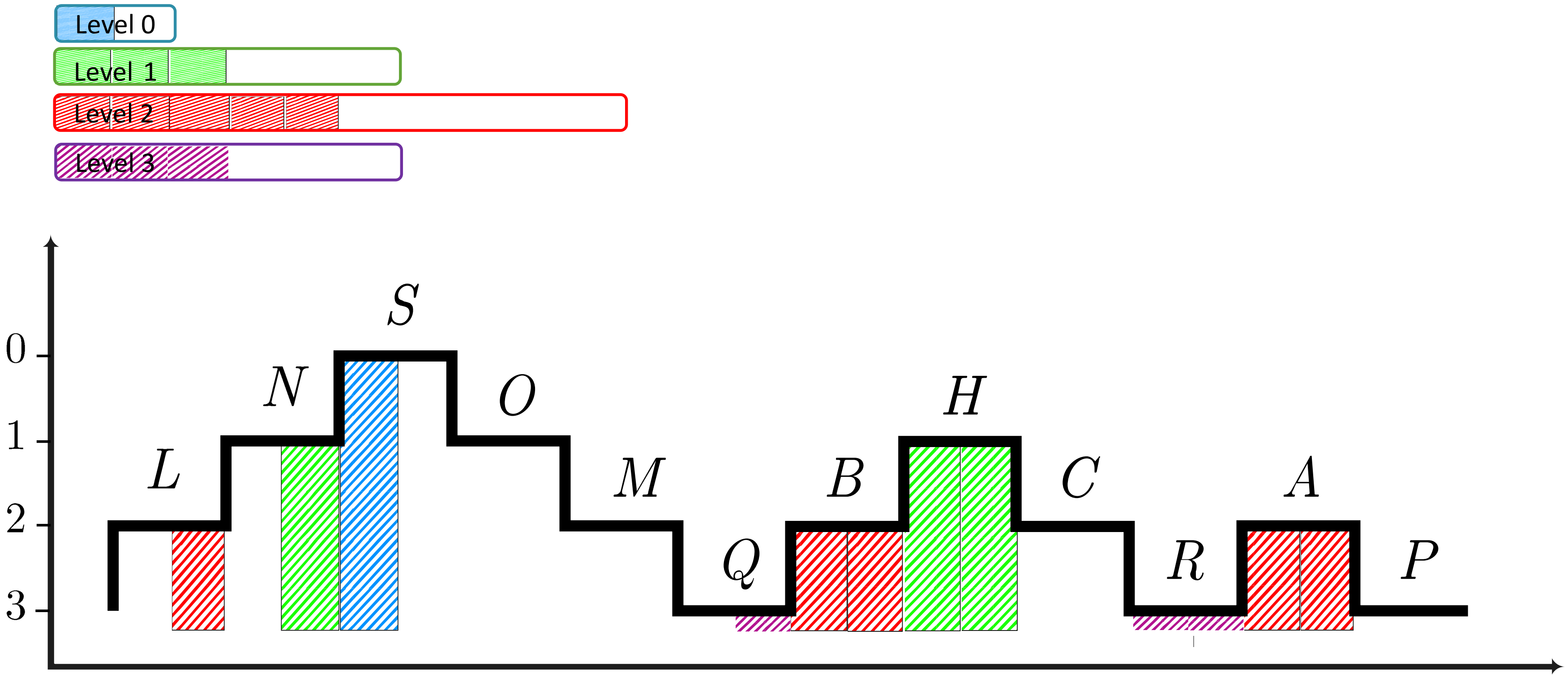
Question Construction





Question Construction





Conclusion

For 3 lies 4 intervals suffice.

General well-shaped structure.

Conditions for well-shapedness preservation.

Future Work

Close the gap between the bound for $e = 0,1,2,3$ and the general case $e > 3$

Non perfect strategies: more questions with less intervals per question.

Analysis of the average case.


Thank you for your attention!


Questions?

Questions?

(Also non 4-intervals questions are allowed)